$$= \left[x\text{Log}\left(1+\frac{1}{x}\right)\right]_{\lambda}^{\alpha} + [\text{Log}(x+1)]_{\lambda}^{\alpha} + \left[-\text{Log}x - \frac{1}{4x}\right]_{\lambda}^{\alpha}$$
$$= (\alpha+1)\text{Log}\left(1+\frac{1}{\alpha}\right) - (\lambda+1)\text{Log}\left(1+\frac{1}{\lambda}\right) + \frac{1}{4}\left(\frac{1}{\lambda} - \frac{1}{\alpha}\right).$$

b- $\lim\limits_{\lambda\to+\infty} A(\lambda)$

$$= (\alpha+1)\text{Log}(1+\tfrac{1}{\alpha}) - \lim_{\lambda\to+\infty}\left[(\lambda+1)\text{Log}(1+\tfrac{1}{\lambda}) - \frac{1}{4}(\tfrac{1}{\lambda}-\tfrac{1}{\alpha})\right]$$
$$=(\alpha+1)\text{Log}(1+\tfrac{1}{\alpha}) - \lim_{\lambda\to+\infty}\lambda\text{Log}(1+\tfrac{1}{\lambda}) - \lim_{\lambda\to+\infty}\left[\underbrace{\text{Log}(1+\tfrac{1}{\lambda})}-\frac{1}{4}(\tfrac{1}{\lambda}-\tfrac{1}{\alpha})\right]$$
$$=(\alpha+1)\text{Log}(1+\tfrac{1}{\alpha}) - \lim_{T\to0^+}\frac{\text{Log}(1+T)}{T} - \tfrac{1}{4\alpha} \text{ avec } T=\tfrac{1}{\lambda}\to 0^+ \text{ qd } \lambda\to+\infty$$
$$=(\alpha+1)\text{Log}(1+\tfrac{1}{\alpha}) - 1 - \tfrac{1}{4\alpha}.$$

B-1/ On a $f(x) \le 0; \forall x \ge 1$

$$\Rightarrow \forall n \in \text{IN}^*; f(n) = \text{Log}\left(1+\frac{1}{n}\right) - \frac{1}{n} + \frac{1}{4n^2} \le 0$$
$$\Leftrightarrow \forall n \in \text{IN}^*; \text{Log}\left(\frac{n+1}{n}\right) \le \frac{1}{n} - \frac{1}{4n^2}$$
$$\Leftrightarrow \forall n \in \text{IN}^*; \text{Log}(n+1) - \text{Log}(n) \le \frac{1}{n} - \frac{1}{4n^2} \qquad (1)$$

$$* \quad \forall n \in \text{IN}^*; \text{Log}\left(\frac{n+1}{n}\right) \le \frac{1}{n} - \frac{1}{4n^2}$$
$$\Leftrightarrow \forall n \in \text{IN}^*; n\text{Log}\left(\frac{n+1}{n}\right) \le 1 - \frac{1}{4n}$$
$$\Leftrightarrow \forall n \in \text{IN}^*; \text{Log}\left[\left(1+\frac{1}{n}\right)^n\right] \le 1 - \frac{1}{4n}.$$

D'où $\forall n \in \text{IN}^*; \left(1+\frac{1}{n}\right)^n \le e^{1-\frac{1}{4n}}$.

2/ $\cdot \dfrac{u_{n+1}}{u_n} = \dfrac{\frac{(n+1)^{n+1}e^{-n-1}}{(n+1)!}}{\frac{n^n e^{-n}}{n!}} = \dfrac{(n+1)\times(n+1)^n e^{-n-1}}{(n+1)\times n!} \times \dfrac{n!}{n^n e^{-n}}$

$$= \frac{(n+1)^n}{n^n}e^{-1} = \left(1+\frac{1}{n}\right)^n e^{-1}.$$

Or on vient de montrer que $\forall n \in \text{IN}^*; \left(1+\frac{1}{n}\right)^n \le e^{1-\frac{1}{4n}}$

$$\Rightarrow \forall n \in \text{IN}^*; \left(1+\frac{1}{n}\right)^n e^{-1} \le e^{-\frac{1}{4n}}$$

Par suite $\dfrac{u_{n+1}}{u_n} \le e^{-\frac{1}{4n}} \Leftrightarrow u_{n+1} \le u_n e^{-\frac{1}{4n}}$ (car $u_n > 0$)

$$0 < u_2 \le u_1 e^{-\frac{1}{4\times1}}$$
$$0 < u_3 \le u_2 e^{-\frac{1}{4\times2}}$$
$$0 < u_4 \le u_3 e^{-\frac{1}{4\times3}}$$
.....